Generalized Vaidya Solutions

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A large family of solutions, representing, in general, spherically symmetric Type II fluid, is presented, which includes most of the known solutions to the Einstein field equations, such as, the monopole-de Sitter-charged Vaidya ones.

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In 1951, Vaidya [1] found a solution that represents an imploding (exploding) null dust fluid with spherical symmetry. Since then, the solution has been intensively studied in gravitational collapse [2]. In particular, Papapetrou [3] first showed that this solution can give rise to the formation of naked singularities, and thus provides one of the earlier counterexamples to the cosmic censorship conjecture [4]. Later, the solution was generalized to the charged case [5]. The charged Vaidya solution soon attracted lot of attention and has been studied in various situations. For example, Sullivan and Israel [6] used it to study the thermodynamics of black holes, and Kaminaga [7] used it as a classical model for the geometry of evaporating charged black holes, while Lake and Zannias [8] studied the self-similar case and found that, similar to the uncharged case, naked singularities can be also formed from gravitational collapse. Quite recently, Husian [9] further generalized the Vaidya solution to a null fluid with a particular equation of state. Husian's solutions have been lately used as the formation of black holes with short hair [10].

In this Letter we shall generalize the Vaidya solution to a more general case, which include most of the known solutions to the Einstein field equations, such as, the monopole-de Sitter-charged Vaidya solutions, and the Husian solutions. The generalization comes from the observation that the energy-momentum tensor (EMT) is linear in terms of the mass function. As a result, the linear superposition of particular solutions is also a solution of the Einstein field equations. To show this, let us begin with the general spherically symmetric line element [11]

$$
\begin{aligned}
ds^2 = & -e^{2\psi(v,r)}\left[1-\frac{2m(v,r)}{r}\right]dv^2 + 2\epsilon e^{\psi(v,r)}dvdr \\
& + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \quad (\epsilon = \pm 1),
\end{aligned} \tag{1}
$$

where $m(v,r)$ is usually called the mass function, and related to the gravitational energy within a given radius r [8,12]. When $\epsilon = +1$, the null coordinate v represents the Eddington advanced time, in which r is decreasing towards the future along a ray $v = Const.$ (ingoing), while when $\epsilon = -1$, it represents the Eddington retarded time, in which r is increasing towards the future along a ray $v = Const.$ (outgoing).

In the following, we shall consider the particular case where $\psi(v,r) = 0$. Then, the non-vanishing components of the Einstein tensor are given by,

$$
G^0_0 = G^1_1 = -\frac{2m'(v,r)}{r^2}, \;\; G^1_0 = \frac{2\dot{m}(v,r)}{r^2}, \;\; G^2_2 = G^3_3 = -\frac{m''(v,r)}{r}, \tag{2}
$$

where $\{x^\mu\} = \{v, r, \theta, \varphi\}$, $(\mu = 0, 1, 2, 3)$, and

$$
\dot{m}(v,r) \equiv \frac{\partial m(v,r)}{\partial v}, \qquad m'(v,r) \equiv \frac{\partial m(v,r)}{\partial r}.
$$

Combining Eq.(2) with the Einstein field equations $G_{\mu\nu} = \kappa T_{\mu\nu}$, we find that the corresponding EMT can be written in the form [9]

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$$T_{\mu\nu} = T^{(n)}_{\mu\nu} + T^{(m)}_{\mu\nu}, \tag{3}$$

where

$$\begin{aligned} T^{(n)}_{\mu\nu} &= \mu l_\mu l_\nu, \\ T^{(m)}_{\mu\nu} &= (\rho + P)\left(l_\mu n_\nu + l_\nu n_\mu\right) + P g_{\mu\nu}, \end{aligned} \tag{4}$$

and

$$\mu = \frac{2\epsilon \dot{m}(v,r)}{\kappa r^2}, \quad \rho = \frac{2m'(v,r)}{\kappa r^2}, \quad P = -\frac{m''(v,r)}{\kappa r}, \tag{5}$$

with l_μ and n_μ being two null vectors,

$$\begin{aligned} l_\mu = \delta^0_\mu, \quad n_\mu &= \frac{1}{2}\left[1 - \frac{2m(v,r)}{r}\right]\delta^0_\mu - \epsilon\delta^1_\mu, \\ l_\lambda l^\lambda = n_\lambda n^\lambda = 0, \quad l_\lambda n^\lambda &= -1. \end{aligned} \tag{6}$$

The part of the EMT, $T^{(n)}_{\mu\nu}$, can be considered as the component of the matter field that moves along the null hypersurfaces $v = Const$. In particular, when $\rho = P = 0$, the solutions reduce to the Vaidya solution with $m = m(v)$. Therefore, for the general case we consider the EMT of Eq.(3) as a generalization of the Vaidya solution.

Projecting the EMT of Eq.(3) to the orthonormal basis, defined by the four vectors,

$$E^\mu_{(0)} = \frac{l_\mu + n_\mu}{\sqrt{2}}, \quad E^\mu_{(1)} = \frac{l_\mu - n_\mu}{\sqrt{2}}, \quad E^\mu_{(2)} = \frac{1}{r}\delta^\mu_2, \quad E^\mu_{(3)} = \frac{1}{r\sin\theta}\delta^\mu_3, \tag{7}$$

we find that

$$\left[T_{(a)(b)}\right] = \begin{bmatrix} \frac{\mu}{2} + \rho & \frac{\mu}{2} & 0 & 0 \\ \frac{\mu}{2} & \frac{\mu}{2} - \rho & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{bmatrix}, \tag{8}$$

which in general belongs to the Type II fluids defined in [13]. The null vector l^μ is a double null eigenvector of the EMT. For this type of fluids, the energy conditions are the following [13]:

a) *The weak and strong energy conditions*:

$$\mu \geq 0, \quad \rho \geq 0, \quad P \geq 0, \ (\mu \neq 0). \tag{9}$$

b) *The dominant energy condition*:

$$\mu \geq 0, \quad \rho \geq P \geq 0, \ (\mu \neq 0). \tag{10}$$

Clearly, by properly choosing the mass function $m(v,r)$, these conditions can be satisfied. In particular, when $m = m(v)$, as we mentioned previously, the solutions reduce to the Vaidya solution, and the energy conditions (weak, strong, and dominant) all reduce to $\mu \geq 0$, while when $m = m(r)$, we have $\mu = 0$, and the matter field degenerates to type I fluid [13]. In the latter case, the energy conditions become:

c) *The weak energy condition*:

$$\rho \geq 0, \quad P + \rho \geq 0, \ \ (\mu = 0). \tag{11}$$

d) *The strong energy condition*:

$$\rho + P \geq 0, \quad P \geq 0, \ \ (\mu = 0). \tag{12}$$

e) *The dominant energy condition*:

$$\rho \geq 0, \quad -\rho \leq P \leq \rho \ \ (\mu = 0). \tag{13}$$

Without loss of generality, we expand $m(v,r)$ in the powers of r,

$$m(v,r) = \sum_{n=-\infty}^{+\infty} a_n(v) r^n, \tag{14}$$

where $a_n(v)$ are arbitrary functions of v only. Note that the sum of the above expression should be understood as an integral, when the "spectrum" index n is continuous. Substituting it into Eq.(5), we find

$$\mu = \frac{2\epsilon}{\kappa} \sum_{n=-\infty}^{+\infty} \dot{a}_n(v) r^{n-2}, \quad \rho = \frac{2}{\kappa} \sum_{n=-\infty}^{+\infty} n a_n(v) r^{n-3},$$
$$P = -\frac{1}{\kappa} \sum_{n=-\infty}^{+\infty} n(n-1) a_n(v) r^{n-3}. \tag{15}$$

The above solutions include most of the known solutions of the Einstein field equations with spherical symmetry:

i) **The monopole solution** [14]: If we choose the functions $a_n(v)$ such that

$$a_n(v) = \begin{cases} \frac{a}{2}, & n = 1, \\ 0, & n \neq 1, \end{cases} \tag{16}$$

where a is an arbitrary constant, then we find

$$m(v,r) = \frac{ar}{2},$$
$$\rho = \frac{a}{\kappa r^2}, \quad \mu = P = 0. \tag{17}$$

Clearly, in this case the matter field is type I and satisfies all the three energy conditions (11) - (12) as long as $a > 0$. The corresponding solution can be identified as representing the gravitational field of a monopole [14] (see also [15]).

ii) **The de Sitter and Anti-de Sitter solutions**: If the functions $a_n(v)$ are chosen such that

$$a_n(v) = \begin{cases} \frac{\Lambda}{6}, & n = 3, \\ 0, & n \neq 3, \end{cases} \tag{18}$$

we find that

$$m(v,r) = \frac{\Lambda}{6} r^3,$$
$$\rho = -P = \frac{\Lambda}{\kappa}, \quad \mu = 0, \tag{19}$$

and that

$$T_{\mu\nu} = -\frac{\Lambda}{\kappa} g_{\mu\nu}. \tag{20}$$

This corresponds to the de Sitter solutions for $\Lambda > 0$, and to Anti-de Sitter solution for $\Lambda < 0$, where Λ is the cosmological constant.

iii) **The charged Vaidya solution**: To obtain the charged Vaidya solution, we shall choose the functions $a_n(v)$ such that,

$$a_n(v) = \begin{cases} f(v), & n = 0, \\ -\frac{q^2(v)}{2}, & n = -1, \\ 0, & n \neq 0, -1, \end{cases} \tag{21}$$

where the two arbitrary functions $f(v)$ and $q(v)$ represent, respectively, the mass and electric charge at the advanced (retarded) time v. Inserting the above expression into Eq.(15), we find that

$$m(v,r) = f(v) - \frac{q^2(v)}{2r},$$
$$\mu = \frac{2\epsilon}{\kappa r^3} \left[r\dot{f}(v) - q(v)\dot{q}(v) \right],$$
$$\rho = P = \frac{q^2(v)}{\kappa r^4}. \tag{22}$$

This is the well-known charged Vaidya solution. $T^{(n)}_{\mu\nu}$ corresponds to the EMT of the Vaidya null fluid, and $T^{(m)}_{\mu\nu}$ to the electromagnetic field, $F_{\mu\nu}$, given by,

$$F_{\mu\nu} = \frac{q(v)}{r^2}(\delta^0_\mu\delta^1_\nu - \delta^1_\mu\delta^0_\nu). \tag{23}$$

From Eq.(22) we can see that the condition $\mu \geq 0$ gives the main restriction on the choice of the functions $f(v)$ and $q(v)$. In particular, if $df/dq > 0$, we can see that there always exists a critical radius r_c such that when $r < r_c$, we have $\mu < 0$, where

$$r_c = q(v)\frac{\dot{q}(v)}{\dot{f}(v)}. \tag{24}$$

Thus, in this case the energy conditions are always violated. However, a closer investigation of the equation of motion for the massless charged particles that consist of the charged null fluid showed that in this case the hypersurface $r = r_c$ is actually a vanishing point [16]. In the imploding case ($\epsilon = +1$), for example, due to the repulsive Lorentz force, the 4-momenta of the particles vanish exactly on $r = r_c$. Afterwards, the Lorentz force will push the particles to move outwards. Therefore, in realistic situations the particles cannot get into the region $r < r_c$, whereby the energy conditions are preserved [16].

iv) **The Husian solutions**: If we choose the functions $a_n(v)$ such that

$$a_n(v) = \begin{cases} f(v), & n = 0, \\ -\frac{g(v)}{2k-1}, & n = 2k-1\ (k \neq 1/2), \\ 0, & n \neq 0, 2k-1, \end{cases} \tag{25}$$

where $f(v)$ and $g(v)$ are two arbitrary functions, and k is a constant, then we find that

$$\begin{aligned} m(v,r) &= f(v) - \frac{g(v)}{(2k-1)r^{2k-1}}, \\ \mu &= \frac{2\epsilon}{\kappa r^2}\left[\dot{f}(v) - \frac{\dot{g}(v)}{(2k-1)r^{2k-1}}\right], \\ P = k\rho &= \frac{2kg(v)}{\kappa r^{2k+2}}. \end{aligned} \tag{26}$$

This is the solution first found by Husian by imposing the equation of state $P = k\rho$ [9]. When $k = 1$, they reduce to the charged Vaidya solution. Similar to the latter case, now the condition $\mu \geq 0$ also gives the main restriction on the choice of the functions $f(v)$ and $g(v)$, especially for the case where $df/dg > 0$. However, one may follow Ori [16] to argue that the hypersurface

$$r = r_c = \left[(2k-1)^{-1}\frac{dg}{df}\right]^{\frac{1}{2k-1}},$$

is also a turning point, although we have not been able to show this explicitly. But the following considerations indeed support this point of view. Following [10], we can cast $T^{(m)}_{\mu\nu}$ into the form of a *generalized* electromagnetic field,

$$T^{(m)}_{\mu\nu} = \frac{2}{\kappa}\left(F_{\mu\lambda}F_\nu^{\ \lambda} - \frac{\alpha}{4}g_{\mu\nu}F_{\lambda\sigma}F^{\lambda\sigma}\right), \tag{27}$$

where $\alpha = 2/(1+k)$, and $F_{\mu\nu}$ can be considered as the generalized electromagnetic field, given by,

$$F_{\mu\nu} = \frac{[k(1+k)m'(v,r)]^{1/2}}{r}(\delta^0_\mu\delta^1_\nu - \delta^1_\mu\delta^0_\nu), \tag{28}$$

which satisfies the Maxwell field equations,

$$F_{[\mu\nu;\lambda]} = 0, \quad F_{\mu\nu;\lambda}g^{\nu\lambda} = J_\mu, \tag{29}$$

with

$$
\begin{aligned}
J_\mu &= J_0\delta^0_\mu + J_1\delta^1_\mu,\\
J_0 &= \frac{2\delta q^{k+1}(v)}{r^{3(k+1)}}\left\{k\dot{q}(v)r^{2(k+1)}\right.\\
&\quad \left.+(1-k)rq(v)[q^{2k}(v) - 2f(v)r^{2k-1} + r^{2k}]\right\},\\
J_1 &= -\frac{2\delta(1-k)q^k(v)}{r^{k+2}},\\
g(v) &= \frac{(2k-1)q^{2k}(v)}{2}, (k \neq 1/2),
\end{aligned}
\tag{30}
$$

where $\delta \equiv [k(1+k)(2k-1)/2]^{1/2}$. When f and g are constants, from Eq.(26) we have $\mu = 0$. Then, the solutions degenerate to type I solutions, and the energy conditions (11) - (13) become, respectively, $g \geq 0,\ k \geq -1$ for the weak energy conditon, $g \geq 0,\ k \geq 0$ or $g \leq 0,\ k \leq -1$ for the strong energy condition, and $g \geq 0,\ -1 \leq k \leq +1$ for the dominant energy condition. Note that when $k > 1$, the "supercharge" q has no contribution to the surface intergral at spatial infinity due to the rapid fall off (r^{-2k}) in the metric coefficients. Therefore, it acts like short hair [10]. However, the existence of this kind of hairs can be limited by the dominant energy condition.

Note that the functions $\mu,\ \rho$ and P are linear in terms of the derivatives of $m(v,r)$. Thus, the linear superposition of Cases i) - iv) is also a solution to the Einstein field equations. In particular, the combination,

$$
m(v,r) = \frac{ar}{2} + \frac{\Lambda}{6}r^3 + f(v) - \frac{q^2(v)}{2r}, \tag{31}
$$

would represent the monopole-de Sitter-charged Vaidya solutions. Obviously, by properly choosing the functions $a_n(v)$, one can obtain as many solutions as wanted. The physical and mathematical properties of these solutions will be studied somewhere else.

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